

z^o -ideals and zero sets in pointfree topology

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Abstract

The main purpose of this note is to introduce and study z^o -ideals of the ring $\mathcal{RL}$ of real-valued continuous functions on L in terms of zero sets in pointfree topology. For a co z -dense and completely regular frame L , it is shown that every ideal in $\mathcal{RL}$ consisting of zero divisors is contained in a z^o -ideal. Basically disconnected frames and P -frames are characterized in terms of z^o -ideals. We characterize frames L for which strongly z -ideals and z^o -ideals coincide in $\mathcal{RL}$. We also investigate frames L such that the sum of any two z^o -ideals in $\mathcal{RL}$ is either $\mathcal{RL}$ or a z^o -ideal

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1 Introduction

We clarify from the start that, throughout, L denotes a frame, ΣL denotes the set of prime elements of L , and $\mathcal{RL}$ denotes the ring of real-valued continuous functions on L .

Let $C(X)$ denotes the ring of real-valued continuous functions on a completely regular Hausdorff space X . Recall [2] that:

- (a) If I is an ideal in $C(X)$ consisting of zero divisors, then I is contained in a z^o -ideal.
- (b) The sum of z^o -ideals in $C(X)$, where X is an almost P -space, is either a z^o -ideal or $C(X)$.
- (c) Every z -ideal in $C(X)$ is a z^o -ideal if and only if X is an almost P -space.

One of the main aims of this note is to develop these results to the more general setting of pointfree topology, that is, frames.

We recall some of the definitions and results on frames and the pointfree version of the ring of continuous real-valued functions that we shall need and refer to [10], [4], and [9] for the frame-theoretic concepts, pointfree function rings, and ring of continuous functions, respectively.

A complete lattice L in which the distributive law $a \wedge \bigvee T = \bigvee \{a \wedge t : t \in T\}$ holds for all $a \in L$ and $T \subseteq L$ is called a *frame*. The top element and the bottom element of L is denoted by $\top$ and $\perp$ respectively. A *frame homomorphism* (or *frame map*) is a map between frames which preserves finite meets, including the top element, and arbitrary joins, including the bottom element. L is said to be *regular* if $a = \bigvee \{x \in L \mid x \prec a\}$ for each $a \in L$, where $x \prec a$ means that $x^* \vee a = \top$, where x^* is the *pseudocomplement* of x . It is said to be *completely regular* if, for each $a \in L$, $a = \bigvee \{x \in L \mid x \prec\prec a\}$, where $x \prec\prec a$ means that there are elements (c_p) indexed by the rational numbers $\mathbb{Q} \cap [0, 1]$ such that $c_0 = x$, $c_1 = a$, and $c_p \prec c_q$ for $p < q$.

An element $p \in L$ is said to be *prime* if $p < \top$ and $a \wedge b \leq p$ implies $a \leq p$ or $b \leq p$. Recall the contravariant *functor* Σ from **Frm** to the category **Top** of topological spaces which assigns to each frame L its *spectrum* ΣL of prime elements with $\Sigma_a = \{p \in \Sigma L \mid a \not\leq p\}$ ($a \in L$) as its open sets.

Recall [4] that the frame $\mathcal{L}(\mathbb{R})$ of reals is obtained by taking the ordered pairs (p, q) of rational numbers as generators. Now for any frame L the *real-valued continuous functions* on L are the homomorphisms $\mathcal{L}(\mathbb{R}) \rightarrow L$. The set $\mathcal{RL}$ of all frame homomorphisms from $\mathcal{L}(\mathbb{R})$ to L has been studied as an f -ring in [4]. For every $r \in \mathbb{R}$, define the constant frame map $\mathbf{r} \in \mathcal{RL}$ by $\mathbf{r}(p, q) = \top$, whenever $p < r < q$, and otherwise $\mathbf{r}(p, q) = \perp$. The *cozero map* is the map $\text{coz} : \mathcal{RL} \rightarrow L$, defined by $\text{coz}(\alpha) = \alpha(-, 0) \vee \alpha(0, -)$. For $A \subseteq \mathcal{RL}$, let $\text{Coz}(A) = \{\text{coz}(\alpha) : \alpha \in A\}$ with the cozero part of a frame L , $\text{Coz}(\mathcal{RL})$, called $\text{Coz}L$.

We also recall necessary notations from [6]. Let $a \in L$ and $\alpha \in \mathcal{RL}$. The sets $\{r \in \mathbb{Q} : \alpha(-, r) \leq a\}$ and $\{s \in \mathbb{Q} : \alpha(s, -) \leq a\}$ are denoted by $L(a, \alpha)$ and $U(a, \alpha)$, respectively. For a frame L , if $p \in \Sigma L$ and $\alpha \in \mathcal{RL}$, then $((p, \alpha), U(p, \alpha))$ is a Dedekind cut for a real number which is denoted by $\tilde{p}(\alpha)$. Throughout, for every $\alpha \in \mathcal{RL}$ we define $\alpha[p] = \tilde{p}(\alpha)$. Now, recall [1, 7] we introduce the pointfree version of zero set $f \in C(X)$ given by $Z(f) = \{x \in X : f(x) = 0\}$, as following:

Let $\alpha \in \mathcal{RL}$. We define $Z(\alpha) = \{p \in \Sigma L : \alpha[p] = 0\}$. Such a set is said to be a zero-set in L . For $A \subseteq \mathcal{RL}$, we write $Z[A]$ to designate the family of zero sets $\{Z(\alpha) : \alpha \in A\}$. The family $Z[\mathcal{RL}]$ of all zero sets in L will also be denoted, for simplicity, by $Z[L]$.

2 Main results

Let A be a commutative ring with identity, $x \in A$ and $S \subseteq A$. We denote the annihilator of S by $\text{Ann}(S)$, and the annihilator of the singleton $\{x\}$ is abbreviated as $\text{Ann}(x)$. Double annihilators will be written as $\text{Ann}^2(S)$, and $\text{Ann}^2(x)$. A frame L is called *coz-dense* if $\Sigma_{\text{coz}(\alpha)} = \emptyset$ implies $\alpha = \mathbf{0}$. Before the main subject is proposed, we begin with the following two propositions which will play a central role in the development.

Proposition 2.1. *L is a coz-dense frame. for every $a \in L$ and $\alpha \in \mathcal{RL}$ the following statements hold.*

1. *If $Z(\alpha) = \Sigma L$, then $\alpha = \mathbf{0}$.*
2. *$Z(\alpha) = \emptyset$ if and only if α is a unit of $\mathcal{RL}$.*
3. *If $\Sigma_a = \emptyset$, then $a = \perp$ for a completely regular frame L .*
4. *If L is a completely regular frame, then $\Sigma_{(\text{coz}(\alpha))^*} = \text{int}_{\Sigma L} Z(\alpha)$.*

Proposition 2.2. *L is a coz-dense frame. For every $\alpha, \beta \in \mathcal{RL}$, we have.*

1. *$Z(\alpha) = Z(\beta)$ if and only if $\text{Ann}(\alpha) = \text{Ann}(\beta)$.*
2. *If L is a completely regular frame L , then $\text{int} Z(\alpha) = \text{int} Z(\beta)$ if and only if $\text{Ann}(\alpha) = \text{Ann}(\beta)$.*

Definition 2.3. An proper ideal I of $\mathcal{RL}$ is called a z^o -ideal if for any $\alpha \in I$ and $\beta \in \mathcal{RL}$, $\text{int} Z(\alpha) = \text{int} Z(\beta)$ implies $\beta \in I$.

An ideal I in $\mathcal{RL}$ is called *strongly z -ideal* if $Z(\alpha) \in Z[I]$ implies that $\alpha \in I$ (see [7]). One checks easily that every maximal ideal in $\mathcal{RL}$ is a strongly z -ideal when L is a coz-dense frame.

Proposition 2.4. *Every z^o -ideal of $\mathcal{RL}$ is strongly z -ideal.*

An proper ideal I of $\mathcal{RL}$ is called a z -ideal if for any $\alpha \in I$ and $\beta \in \mathcal{RL}$, $\text{coz} \alpha = \text{coz} \beta$ implies $\beta \in I$ (see [5]). Since every strongly z -ideal of $\mathcal{RL}$ is a z -ideal, by previous proposition, every z^o -ideal of $\mathcal{RL}$ is a z -ideal.

Example 2.5. We provide some examples of z^o -ideals on $\mathcal{RL}$ in the following.

1. If $S \subseteq \Sigma L$ such that $S = \text{cl}_{\Sigma L} \text{int}_{\Sigma L}(S)$, Then the set $M_S = \{\alpha \in \mathcal{RL} \mid S \subseteq Z(\alpha)\}$ is a z^o -ideal.
2. For every $p \in \Sigma L$, the set $O_p = \{\alpha \in \mathcal{RL} \mid p \in \text{int}_{\Sigma L} Z(\alpha)\}$ is a z^o -ideal.
3. The intersection an arbitrary family of z^o -ideals in $\mathcal{RL}$ is a z^o -ideal.
4. If L is a coz-dense and completely regular frame, then P_α is a z^o -ideal for all $\alpha \in \mathcal{RL}$.

For each $\alpha \in \mathcal{RL}$, let P_α be the intersection of all minimal prime ideals containing α by convention, the intersection of an empty set of ideals is $\mathcal{RL}$. According to what was discussed, the next theorem is now immediate.

Theorem 2.6. *Let L be a coz-dense and completely regular frame. If I is a proper ideal in $\mathcal{RL}$, then the following statements are equivalent.*

1. I is a z^o -ideal in the ring $\mathcal{RL}$.
2. $\Sigma_{(\text{coz}(\alpha))^*} = \Sigma_{(\text{coz}(\beta))^*}$ and $\alpha \in I$ imply that $\beta \in I$.
3. $(\text{coz}(\alpha))^* = (\text{coz}(\beta))^*$ and $\alpha \in I$ imply that $\beta \in I$.
4. $P_\alpha = P_\beta$ and $\alpha \in I$ imply that $\beta \in I$.
5. $\text{Ann}(\alpha) = \text{Ann}(\beta)$ and $\alpha \in I$ imply that $\beta \in I$.
6. $\alpha \in I$ implies that $\text{Ann}(\text{Ann}(\alpha)) \subseteq I$.
7. If $\text{Ann}(S) \subset \text{Ann}(\beta)$ and $S \subset I$ is a finite set, then $\beta \in I$.

An ideal I of $\mathcal{RL}$ is called a d -ideal if for any $\alpha \in I$ and $\beta \in \mathcal{RL}$, $\text{Ann}(\alpha) = \text{Ann}(\beta)$ implies $\beta \in I$ ([5]).

Corollary 2.7. *Let L be a coz-dense and completely regular frame and I be a proper ideal in $\mathcal{RL}$. Then I is a z^o -ideal if and only if I is a d -ideal*

Corollary 2.8. *Let L be a coz-dense and completely regular frame. For $\alpha \in \mathcal{RL}$,*

$$P_\alpha = \{\beta \in \mathcal{RL} \mid \text{int}_{\Sigma L}(Z(\alpha)) \subseteq \text{int}_{\Sigma L}(Z(\beta))\} = \{\beta \in \mathcal{RL} \mid \text{Ann}(\alpha) \subseteq \text{Ann}(\beta)\}.$$

Clearly, P_α is a z^o -ideal which is said to be a basic z^o -ideal.

Theorem 2.9. *Let L be a coz-dense and completely regular frame. If I is an ideal in $\mathcal{RL}$ consisting of zero divisors, then I is contained in a z^o -ideal.*

Corollary 2.10. *For a coz-dense and completely regular frame L , every maximal ideal in $\mathcal{RL}$ consisting only of zero divisors is a z^o -ideal.*

Corollary 2.11. *Let L be a coz-dense and completely regular frame. If I is an ideal in $\mathcal{RL}$ consisting of zero divisors, then there is the smallest z^o -ideal containing I and also there is a maximal z^o -ideal containing I which is also a prime z^o -ideal.*

A frame L is said to be *basically disconnected*, if $a^* \vee a^{**} = \top$ for all $a \in \text{Coz}L$ (see [3]). Also, a frame L is said to be *weakly spatial*, if $\Sigma_a \neq \Sigma_\top$, whenever $a < \top$. every weakly spatial frame is coz-dense (see [8]).

Proposition 2.12. *If L is basically disconnected, then $Z(\alpha) = \text{cl}_{\Sigma L} \text{int}_{\Sigma L}(Z(\alpha))$. If L is a weakly spatial completely regular frame, then the converse is also true.*

Now we give an algebraic characterization of basically disconnected frames in terms of z^o -ideals.

Theorem 2.13. *For a coz-dense and completely regular frame L , Every basic z^o -ideal in $\mathcal{RL}$ is principal if and only if L is basically disconnected.*

A frame L is an almost P -frame if every cozero element is regular. This means that the only dense cozero element of L is $\top$, or equivalently every element of $\mathcal{RL}$ is either a unit or a zero divisor (see [5]).

Proposition 2.14. *For a coz-dense and completely regular frame L , then the following equivalent.*

1. L is almost- P .
2. Every non-empty zero set of L has a non-empty interior.
3. For every $\alpha \in \mathcal{RL}$, $Z(\alpha)$ is a regular closed set, i.e., $Z(\alpha) = \text{cl}_{\Sigma L} \text{int}_{\Sigma L}(Z(\alpha))$.

In what follows, we are going to investigate frames L such that the sum of two z^o -ideals in $\mathcal{RL}$ either is a z^o -ideal or equals $\mathcal{RL}$.

Theorem 2.15. *For a coz-dense and completely regular frame L , if L is a basically disconnected frame, then the sum of two z^o -ideals is either a z^o -ideal or $\mathcal{RL}$.*

The next theorem, which is an algebraic characterization of almost P -frames, instantly demonstrates that the sum of z^o -ideals in $\mathcal{R}L$ is either a z^o -ideal or $\mathcal{R}L$, whenever L is an almost P -frame.

Theorem 2.16. *For a co z -dense and completely regular frame L , the following are equivalent.*

1. L is an almost P -frame.
2. Every strongly z -ideal in $\mathcal{R}L$ is a z^o -ideal.
3. Every maximal ideal (prime strongly z -ideal) in $\mathcal{R}L$ is a z^o -ideal.
4. Every maximal ideal in $\mathcal{R}L$ consists entirely of zero divisors.
5. The sum of any two ideals consisting of zero divisors either is $\mathcal{R}L$ or consists of zero divisors.
6. For each nonunit element $\alpha \in \mathcal{R}L$ there exists a nonzero $\beta \in \mathcal{R}L$ with $P_\alpha \subset \text{Ann}(\beta)$.

A frame L is said to be P -frame, if $\text{co}z\alpha \vee (\text{co}z\alpha)^* = \top$ for all $\alpha \in \mathcal{R}L$ (see [3]).

Proposition 2.17. *Let L be a weakly spatial and completely regular frame. Then L is P -frame if and only if every zero set of L is open.*

Theorem 2.18. *For a weakly spatial and completely regular frame L , the following are equivalent.*

1. L is an P -frame.
2. Every ideal in $\mathcal{R}L$ consisting of zero divisors is a z^o -ideal.
3. Every nonunit element of $\mathcal{R}L$ is a zero divisor and P_α is a principal ideal in $\mathcal{R}L$ for all $\alpha \in \mathcal{R}L$.

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